## SECTION 6.5 (12.1, 12.3): ARC LENGTH

Suppose the curve $C$ is described as the graph of a function $y = f(x)$ over the interval $[a, b]$.

If the derivative of $f$, $f'$, is continuous on $[a, b]$,[1] we define the **arc length** of $C$ to be the value of the integral:

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

**NOTE:** The differential $ds = \sqrt{1 + [f'(x)]^2}\, dx$ is called the **arc length differential** and we write:

$$\text{arc length} = s = \int_C ds = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

**EXAMPLE 1:** Find the arc length of the following graphs.

1. $y = \ln|\sec(x)|$ over $\left[0, \frac{\pi}{3}\right]$.

   Ans: $ds = \sqrt{1 + \tan^2(x)}\, dx = ... = \sec(x)\, dx$ so $s = \displaystyle\int_0^{\frac{\pi}{3}} \sec(x)\, dx = ... = \ln\left(2 + \sqrt{3}\right)$ units

2. $y = x^2$ over $\left[0, \frac{1}{2}\right]$.

   Ans: $ds = \sqrt{1 + (2x)^2}\, dx = \sqrt{1 + 4x^2}\, dx$ so $s = \displaystyle\int_0^{\frac{1}{2}} \sqrt{1 + 4x^2}\, dx = ... = \frac{\sqrt{2}}{4} + \frac{1}{4}\ln\left(\sqrt{2} + 1\right)$ units

3. $y = \dfrac{x^3}{3} + \dfrac{1}{4x}$ over $[1, 3]$.

   Ans: $ds = \sqrt{1 + \left(x^2 - \dfrac{1}{4x^2}\right)^2}\, dx = ... = \dfrac{4x^4 + 1}{x^2}\, dx$ so $s = \displaystyle\int_1^3 \frac{4x^4 + 1}{x^2}\, dx = ... = \frac{53}{6}$ units

4. $y = e^{\frac{x}{2}} + e^{-\frac{x}{2}}$ over $[0, \ln(2)]$.

   Ans: $ds = \sqrt{1 + \left(\dfrac{e^{\frac{x}{2}} - e^{-\frac{x}{2}}}{2}\right)^2}\, dx = ... = \dfrac{1}{2}\left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right) dx$ so $s = \displaystyle\frac{1}{2}\int_0^{\ln(2)} \left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right) dx = ... = \frac{\sqrt{2}}{2}$ units

   **NOTE:** If you're a fan of hyperbolic functions, $y = 2\cosh\left(\dfrac{x}{2}\right)$ and you can use hyperbolic identities . . .

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[1] Such a function is said to be **smooth** or **rectifiable** on $[a, b]$

**EXAMPLE 2:** Find the arc length of $y = 3x^{\frac{2}{3}}$:

1. From $(1, 3)$ to $(8, 12)$.

   Ans: $ds = \sqrt{1 + \left(2x^{-\frac{1}{3}}\right)^2}\, dx = \ldots = x^{-\frac{1}{3}}\sqrt{x^{\frac{2}{3}} + 4}\, dx$ so $s = \int_1^8 x^{-\frac{1}{3}}\sqrt{x^{\frac{2}{3}} + 4}\, dx = \ldots = 16\sqrt{2} - 5\sqrt{5}$ units

2. From $(0, 0)$ to $(8, 12)$

   Ans: $y = 3x^{\frac{2}{3}}$ is not differentiable at $(0, 0)$ so we reframe the problem as $x = \left(\frac{y}{3}\right)^{\frac{3}{2}}$, for $0 \le y \le 12$:

   $$ds = \sqrt{1 + \left(\frac{1}{2}\left(\frac{y}{3}\right)^{\frac{1}{2}}\right)^2}\, dy = \ldots = \sqrt{1 + \frac{y}{12}}\, dy \text{ so } s = \int_0^{12} \sqrt{1 + \frac{y}{12}}\, dy = \ldots = 16\sqrt{2} - 8 \text{ units}$$

   **NOTE:** We **could** have integrated $s = \int_0^8 x^{-\frac{1}{3}}\sqrt{x^{\frac{2}{3}} + 4}\, dx$ keeping in mind this is an improper integral:

   $$s = \int_0^8 x^{-\frac{1}{3}}\sqrt{x^{\frac{2}{3}} + 4}\, dx = \lim_{b \to 0^+} \int_b^8 x^{-\frac{1}{3}}\sqrt{x^{\frac{2}{3}} + 4}\, dx = \ldots = 16\sqrt{2} - 8 \text{ units}$$

**ARC LENGTH FOR PARAMETRIC CURVES:**

If $(x(t), y(t))$ for $t$ in $[a, b]$ is a smooth parameterization of $C$ then: $ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$ so that

$$s = \int_C ds = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

**EXAMPLE 3:** Find the arc length of one arch of the cycloid $\{x = t - \sin(t), y = 1 - \cos(t)$.

Ans: $ds = \sqrt{(1 - \cos(t))^2 + (\sin(t))^2}\, dt = \ldots = \sqrt{2 - 2\cos(t)}\, dt = \sqrt{4\sin^2\left(\frac{t}{2}\right)}\, dt = 2\left|\sin\left(\frac{t}{2}\right)\right|\, dt$ so

$$s = \int_0^{2\pi} 2\left|\sin\left(\frac{t}{2}\right)\right|\, dt = \ldots = 8 \text{ units}$$

**ARC LENGTH FOR POLAR CURVES:**

If $C$ is the graph of a smooth polar curve $r = f(\theta)$ for $\theta$ in $[\alpha, \beta]$, then $ds = \sqrt{(r(\theta))^2 + (r'(\theta))^2}\, d\theta$ and:

$$s = \int_C ds = \int_\alpha^\beta \sqrt{(r(\theta))^2 + (r'(\theta))^2}\, d\theta$$

**EXAMPLE 4:** Find the arc length of the polar curve $r = 1 + \cos(\theta)$.

Ans: $ds = \sqrt{(1 + \cos(\theta))^2 + (-\sin(\theta))^2}\, d\theta = \ldots = \sqrt{2 + 2\cos(\theta)}\, d\theta = \sqrt{4\cos^2\left(\frac{\theta}{2}\right)}\, d\theta = 2\left|\cos\left(\frac{\theta}{2}\right)\right|\, d\theta$ so

$$s = \int_0^{2\pi} 2\left|\cos\left(\frac{\theta}{2}\right)\right|\, d\theta = \ldots = 8 \text{ units}$$

## SECTION 6.6 (12.1): SURFACE AREA OF REVOLUTION

If we rotate a curve $C$ about an axis, we can calculate the resulting surface area of revolution, $S$ as follows:

$$S = \int_C 2\pi y\, ds \quad \text{(rotation about the } x\text{-axis)} \quad \textbf{or} \quad S = \int_C 2\pi x\, ds \quad \text{(rotation about the } y\text{-axis)}$$

The differentials $dS = 2\pi y\, ds$ or $dS = 2\pi x\, ds$ are called the **surface area differentials** and we write: $S = \displaystyle\int_C dS$.

**EXAMPLE 5:** Find the surface area of the following:

1. The surface generated by rotating the graph of $y = x^2$ from $x = 1$ to $x = 2$ about $y$-axis:

   Ans: $dS = 2\pi x\, ds = 2\pi x\sqrt{1+4x^2}\, dx$ so $S = 2\pi \displaystyle\int_1^2 x\sqrt{1+4x^2}\, dx = \ldots = \frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right)$ units$^2$

2. The surface generated by rotating the graph of $y = \sqrt{x}$ from $x = 1$ to $x = 4$ about $x$-axis:

   Ans: $dS = 2\pi y\, ds = 2\pi\sqrt{x}\sqrt{1 + \dfrac{1}{4x}}\, dx = \ldots = 2\pi\sqrt{x + \dfrac{1}{4}}\, dx$ so:

   $S = 2\pi \displaystyle\int_1^4 2\pi\sqrt{x + \frac{1}{4}}\, dx = \ldots = \frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right)$ units$^2$ (compare this with #1)

3. Recall Gabriel's Horn is the object obtained by rotating the graph of $y = \dfrac{1}{x}$, $x \geq 1$ around the $x$-axis.

   Show the surface area of Gabriel's Horn is infinite.

   Ans: $dS = 2\pi y\, ds = 2\pi\dfrac{1}{x}\sqrt{1 + \left(-\frac{1}{x^2}\right)^2}\, dx = 2\pi\dfrac{1}{x}\sqrt{1 + \dfrac{1}{x^4}}\, dx$ so

   $$S = \int_1^\infty 2\pi\frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\, dx \geq \int_1^\infty \frac{1}{x}\, dx = \infty$$

4. Let $R > 0$ and let $C$ be the curve described parametrically as $\{x = R\cos(t), y = R\sin(t)\,,\ 0 \leq t \leq \pi$.

   Find the area of the surface obtained by rotating $C$ about the $x$-axis. Do you recognize the formula?

   Ans: $dS = 2\pi y\, ds = 2\pi R\sin(t)\sqrt{(-R\sin(t))^2 + (R\sin(t))^2}\, dt = \ldots = 2\pi R^2\sin(t)\, dt$ so:

   $$S = 2\pi R^2\int_0^\pi \sin(t)\, dt = 4\pi R^2 \text{ units}^2$$

   which is the formula for the surface area of a sphere of radius $R$!

**HOMEWORK:**

**Arc Length:**
Section 6.5: 9 - 31 odd; Section 12.1: 81 - 87 odd; Section 12.3: 63 - 71 odd

**Surface Area:**
Section 6.6: 7 - 19 odd; Section 12.1: 107, 109